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Interval oscillation criteria for second order partial differential equations with delays☆

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Abstract

In this paper, we present new interval oscillation criteria related to integral averaging technique for second order partial differential equations with delays that are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on whole half-line. Our results are sharper than some previous results and handles the cases which are not covered by known criteria.

MSC: 35B05; 35R10

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1. Introduction

Consider the second order delay partial differential equation

$$\frac{\partial}{\partial t}\left(p(t)\frac{\partial}{\partial t}u(x,t)\right) = a(t)\Delta u(x,t) + \sum_{k=1}^{s} a_k(t)\Delta u(x, t-\rho_k(t))$$
$$- q(t)u(x,t) - \sum_{j=1}^{m} q_j(t)u(x, t-\sigma_j), \quad (x,t) \in \Omega \times R_+ \equiv G, \tag{1.1}$$

where Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$, $R_+ = [0,\infty)$ and $\Delta u(x,t) = \sum_{i=1}^{N} (\partial^2 u(x,t)/\partial x_i^2)$.

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Throughout this paper, we assume that

(H$_1$) $p \in C^1(R_+; (0, \infty))$, $\int^{\infty} \mathrm{d}s/p(s) = \infty$;
(H$_2$) $q, q_j \in C(R_+; R_+)$; $j \in I_m = [1, 2, \cdots, m]$;
(H$_3$) $a, a_k \in C(R_+; R_+)$, $\rho_k \in C(R_+; R_+)$, $\lim_{t\to\infty}(t - \rho_k(t)) = \infty$, σ_j are nonnegative constants, $j \in I_m, k \in I_s$.

Consider the following boundary condition:

$$\frac{\partial u(x,t)}{\partial \gamma} + g(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+, \tag{1.2}$$

where γ is the unit exterior normal vector to $\partial\Omega$ and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega \times R_+$.

Definition 1.1. The solution $u(x,t)$ of the problem (1.1), (1.2) is said to be oscillatory in the domain G if for any positive number μ there exists a point $(x_0, t_0) \in \Omega \times [\mu, \infty)$ such that $u(x_0, t_0) = 0$ holds.

Definition 1.2. We say that a function $H = H(t,s)$ belongs to a function class Ψ, denoted by $H \in \Psi$. If $H \in C(D; R_+)$, where $D = \{(t,s), -\infty < t, s < \infty\}$ which satisfies

$$H(t,t) = 0, \quad H(t,s) > 0 \ \text{ for } t > s, \tag{1.3}$$

and has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)}, \quad \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}, \tag{1.4}$$

where $h_1, h_2 \in L_{\mathrm{loc}}(D; R)$.

Recently, there has been an increase in studying the oscillation for the second order delay partial differential equations. For example, see [1–13]. However, all above mentioned oscillation results involve the integral of p, q and hence require the information of p, q on the entire half-line $[t_0, \infty)$.

In this paper, by using Riccati technique we obtain several new interval criteria for oscillation, that is criteria by the behavior of p, q only on a sequence of subintervals of $[t_0, \infty)$. Our results improve and extend the results of [3–5,8–10].

2. Main results

The following lemmas will be useful in the proof of our main results.

Lemma 2.1 (Li[10]). *Suppose that the differential inequality*

$$(p(t)v'(t))' + q(t)v(t) + \sum_{j=1}^{m} q_j(t)v(t - \sigma_j) \leqslant 0 \tag{2.1}$$

has no eventually positive solutions, where

$$v(t) = \int_{\Omega} u(x,t)\,\mathrm{d}x, \quad t \geqslant t_0. \tag{2.2}$$

Then every solution $u(x,t)$ of the problem (1.1), (1.2) *is oscillatory on G.*

Lemma 2.2. *If $v(t)$ is a solution of* (2.1) *such that $v(t) > 0$, $t \geqslant T_0 - \sigma$, for some $T_0 \geqslant t_0$, where $\sigma = \max_{1\leqslant j\leqslant m}\sigma_j$, then*

$$(p(t)v'(t))' \leqslant 0, \quad v'(t) \geqslant 0, \quad t \geqslant T_0. \tag{2.3}$$

Proof. It follows from (H_2) and (2.1) we have

$$(p(t)v'(t))' \leqslant 0, \quad t \geqslant T_0. \tag{2.4}$$

Here $p(t)v'(t)$ is a decreasing function. We claim that $v'(t) \geqslant 0$ for $t \geqslant T_0$. In fact, if there is a $t_1 > T_0$ such that $v'(t_1) < 0$. It follows from (2.4)

$$v(t) \leqslant v(t_1) + p(t_1)v'(t_1) \int_{t_1}^{t} \frac{\mathrm{d}s}{p(s)}, \quad t \geqslant t_1.$$

Therefore, by (H_1), we have $\lim_{t\to\infty} v(t) = -\infty$, which contradicts the fact that $v(t) > 0$ for $t \geqslant T_0$. This completes the proof. □

Lemma 2.3. *If $v(t)$ is a solution of* (2.1) *such that $v(t) > 0$ on* [c, b) *where* $(c > T_0 - \sigma)$, *for any $j_0 \in I_m$, let*

$$u(t) = \frac{p(t)v'(t)}{v(t-\sigma_{j_0})}, \quad t \in [c, b), \tag{2.5}$$

then for any $H \in \Psi$,

$$\int_c^b \{H(b,s)q_{j_0}(s) - \frac{1}{4} p(s-\sigma_{j_0})h_2^2(b,s)\}\,\mathrm{d}s \leqslant H(b,c)u(c). \tag{2.6}$$

Proof. From (2.1) and (H_2), we obtain

$$(p(t)v'(t))' + q_{j_0}(t)v(t-\sigma_{j_0}) \leqslant 0, \quad t \in [c, b). \tag{2.7}$$

From (2.5) and (2.7), we get

$$u'(t) \leqslant -q_{j_0}(t) - \frac{p(t)v'(t)v'(t-\sigma_{j_0})}{v^2(t-\sigma_{j_0})}, \quad t \in [c, b).$$

Using the fact that $p(t)v'(t)$ is decreasing, we have

$$p(t)v'(t) \leqslant p(t-\sigma_{j_0})v'(t-\sigma_{j_0}), \quad t \in [c, b).$$

Thus,

$$\begin{aligned} u'(t) &\leqslant -q_{j_0}(t) - \frac{1}{p(t-\sigma_{j_0})}\left(\frac{p(t)v'(t)}{v(t-\sigma_{j_0})}\right)^2 \\ &= -q_{j_0}(t) - \frac{1}{p(t-\sigma_{j_0})}u^2(t). \end{aligned} \tag{2.8}$$

Multiplying (2.8), with t replaced by s, by $H(t,s)$ and integrating from c to t for $t \in [c, b)$ and using (1.3) and (1.4), we get

$$\begin{aligned} \int_c^t H(t,s)q_{j_0}(s)\,\mathrm{d}s &\leqslant -\int_c^t H(t,s)u'(s)\,\mathrm{d}s - \int_c^t H(t,s)\frac{u^2(s)}{p(s-\sigma_{j_0})}\,\mathrm{d}s \\ &= H(t,c)u(c) - \int_c^t h_2(t,s)\sqrt{H(t,s)}u(s)\,\mathrm{d}s - \int_c^t H(t,s)\frac{u^2(s)}{p(s-\sigma_{j_0})}\,\mathrm{d}s \\ &= H(t,c)u(c) + \frac{1}{4}\int_c^t p(s-\sigma_{j_0})h_2^2(t,s)\,\mathrm{d}s \\ &\quad - \int_c^t \left\{\sqrt{\frac{H(t,s)}{p(s-\sigma_{j_0})}}u(s) + \frac{1}{2}h_2(t,s)\sqrt{p(s-\sigma_{j_0})}\right\}^2 \mathrm{d}s. \end{aligned}$$

Hence, we have

$$\int_c^t H(t,s)q_{j_0}(s)\,\mathrm{d}s-\frac{1}{4}\int_c^t p(s-\sigma_{j_0})h_2^2(t,s)\,\mathrm{d}s\leqslant H(t,c)u(c).$$

Letting $t\to b^{-0}$ in the above, we obtain (2.6). The proof is completed. □

Lemma 2.4. *If $v(t)$ is a solution of* (2.1) *such that $v(t)>0$ on $(a,c]$, where $a>T_0-\sigma$, for any $j_0\in I_m$, let $u(t)$ defined by* (2.5) *on* (a, c], *then for any $H\in\Psi$,*

$$\int_a^c \{H(s,a)q_{j_0}(s)-\frac{1}{4}p(s-\sigma_{j_0})h_1^2(s,a)\}\,\mathrm{d}s\leqslant -H(c,a)u(c). \tag{2.9}$$

Proof. Similar to the proof of Lemma 2.3, (2.8) holds. We multiply (2.8) by $H(s,t)$, integrate it with respect to s from t to c for $t\in(a,c]$, and use (1.3) and (1.4), we get

$$\begin{aligned}\int_t^c H(s,t)q_{j_0}(s)\,\mathrm{d}s&\leqslant -\int_t^c H(s,t)u'(s)\,\mathrm{d}s-\int_t^c \frac{H(s,t)}{p(s-\sigma_{j_0})}u^2(s)\,\mathrm{d}s\\&=-H(c.t)u(c)+\int_t^c h_1(s,t)\sqrt{H(s,t)}u(s)\,\mathrm{d}s-\int_t^c \frac{H(s,t)}{p(s-\sigma_{j_0})}u^2(s)\,\mathrm{d}s\\&=-H(c,t)u(c)+\frac{1}{4}\int_t^c p(s-\sigma_{j_0})h_1^2(s,t)\,\mathrm{d}s\\&\quad-\int_t^c\left\{\sqrt{\frac{H(s,t)}{p(s-\sigma_{j_0})}}u(s)-\frac{1}{2}\sqrt{p(s-\sigma_{j_0})}h_1(s,t)\right\}^2\mathrm{d}s.\end{aligned}$$

Hence, we have

$$\int_t^c \{H(s,t)q_{j_0}(s)-\frac{1}{4}p(s-\sigma_{j_0})h_1^2(s,t)\}\,\mathrm{d}s\leqslant -H(c,t)u(c).$$

Letting $t\to a^+$ in the above, we obtain (2.9). The proof is complete. □

Lemma 2.5. *If $v(t)$ is a solution of* (2.1) *such that $v(t)>0$ on $[c,b)$, for any $j_0\in I_m$, let*

$$\bar u(t)=F(t)\left\{\frac{p(t)v'(t)}{v(t-\sigma_{j_0})}+p(t-\sigma_{j_0})f(t)\right\},\quad t\in[c,b), \tag{2.10}$$

where $f\in C^1[t_0,\infty)$ and $F(t)=\exp(-2\int_{t_0}^t f(s)\,\mathrm{d}s)$. Then for any $H\in\Psi$,

$$\int_c^b \{H(b,s)\chi(s)-\frac{1}{4}F(s)p(s-\sigma_{j_0})h_2^2(b,s)\}\,\mathrm{d}s\leqslant H(b,c)\bar u(c), \tag{2.11}$$

where

$$\chi(s)=F(s)\{q_{j_0}(s)+p(s-\sigma_{j_0})f^2(s)-[p(s-\sigma_{j_0})f(s)]'\}. \tag{2.12}$$

Proof. From (2.7) and (2.10), we obtain

$$\begin{aligned}\bar u'(t)&\leqslant -2f(t)\bar u(t)\\&\quad+F(t)\left\{-q_{j_0}(t)-\frac{p(t)v'(t)v'(t-\sigma_{j_0})}{v^2(t-\sigma_{j_0})}+(p(t-\sigma_{j_0})f(t))'\right\}.\end{aligned}$$

Using the fact that $p(t)v'(t) \leqslant p(t-\sigma_{j_0})v'(t-\sigma_{j_0})$, we get

$$\begin{aligned}\bar{u}'(t) &\leqslant -2f(t)\bar{u}(t) + F(t)\left\{-q_{j_0}(t) - \frac{1}{p(t-\sigma_{j_0})}\left(\frac{p(t)v'(t)}{v(t-\sigma_{j_0})}\right)^2 + (p(t-\sigma_{j_0})f(t))'\right\} \\ &= -2f(t)\bar{u}(t) + F(t)\left\{-q_{j_0}(t) - \frac{1}{p(t-\sigma_{j_0})}\left(\frac{\bar{u}(t)}{F(t)} - p(t-\sigma_{j_0})f(t)\right)^2 + [p(t-\sigma_{j_0})f(t)]'\right\} \\ &= -\chi(t) - \frac{\bar{u}^2(t)}{p(t-\sigma_{j_0})F(t)}.\end{aligned}$$

The rest of the proof is similar to that of Lemma 2.3 and hence is omitted here. □

Lemma 2.6. *If $v(t)$ is a solution of* (2.1) *such that $v(t) > 0$ on $(a, c]$, for any $j_0 \in I_m$, let $\bar{u}(t)$ be defined by* (2.10), *them for any $H \in \Psi$,*

$$\int_a^c \{H(s,a)\chi(s) - \frac{1}{4}p(s-\sigma_{j_0})F(s)h_1^2(s,a)\}\,\mathrm{d}s \leqslant -H(c,a)\bar{u}(c), \tag{2.13}$$

where $F(s)$, $\chi(s)$ are defined as in Lemma 2.5.

The proof is similar to the proof of Lemma 2.5, hence we omit the details.

The following theorem is an immediate result from Lemmas 2.1 to 2.4.

Theorem 2.1. *Suppose that for any $T \geqslant t_0$, there exist $H \in \Psi$, $j_0 \in I_m$, and $a, b, c \in R$ such that $T \leqslant a < c < b$ and,*

$$\begin{aligned}&\frac{1}{H(c,a)}\int_a^c \{H(s,a)q_{j_0}(s) - \frac{1}{4}p(s-\sigma_{j_0})h_1^2(s,a)\}\,\mathrm{d}s \\ &\quad + \frac{1}{H(b,c)}\int_c^b \{H(b,s)q_{j_0}(s) - \frac{1}{4}p(s-\sigma_{j_0})h_2^2(b,s)\}\,\mathrm{d}s > 0.\end{aligned} \tag{2.14}$$

Then every solution $u(x,t)$ of the problem (1.1), (1.2) *is oscillatory in G.*

proof. By Lemma 2.1 we only will prove that the inequality (2.1) has no eventually positive solutions. Suppose that $v(t)$ is an eventually positive solution of (2.1), without loss of generality, we may assume that $v(t) > 0$, $t \geqslant t_1$. By assumption, we can choose $a, b, c \in R$ such that $t_1 \leqslant a < c < b$, and (2.14) holds. From Lemmas 2.3 and 2.4, we see that both (2.6) and (2.9) hold. By dividing (2.6) and (2.9) by $H(b,c)$ and $H(c,a)$, respectively, and then adding them, we have

$$\begin{aligned}&\frac{1}{H(c,a)}\int_a^c \{H(s,a)q_{j_0}(s) - \frac{1}{4}p(s-\sigma_{j_0})h_1^2(s,a)\}\,\mathrm{d}s \\ &\quad + \frac{1}{H(b,c)}\int_c^b \{H(b,s)q_{j_0}(s) - \frac{1}{4}p(s-\sigma_{j_0})h_2^2(b,s)\}\,\mathrm{d}s \leqslant 0\end{aligned}$$

which contradicts the assumption (2.14) and the proof is complete. □

Theorem 2.2. *If for some $H \in \Psi$, $j_0 \in I_m$, and for each $r \geqslant t_0$,*

$$\limsup_{t\to\infty} \int_r^t \{H(s,r)q_{j_0}(s) - \frac{1}{4}p(s-\sigma_{j_0})h_1^2(s,r)\}\,\mathrm{d}s > 0, \tag{2.15}$$

and

$$\limsup_{t\to\infty} \int_r^t \{H(t,s)q_{j_0}(s) - \frac{1}{4}p(s-\sigma_{j_0})h_2^2(t,s)\}\,\mathrm{d}s > 0. \tag{2.16}$$

Then every solution $u(x,t)$ of the problem (1.1), (1.2) *is oscillatory in G.*

Proof. For any $T \geqslant t_0$, let $a = T$. In (2.15) we choose $r = a$. Then there exists $c > a$ such that

$$\int_a^c \{H(s,a)q_{j_0}(s) - \frac{1}{4}p(s-\sigma_{j_0})h_1^2(s,a)\}\,\mathrm{d}s > 0. \tag{2.17}$$

In (2.16) we choose $r = c$, then there exists $b > c$ such that

$$\int_c^b \{H(b,s)q_{j_0}(s) - \frac{1}{4}p(s-\sigma_{j_0})h_2^2(b,s)\}\,\mathrm{d}s > 0. \tag{2.18}$$

Combining (2.17) and (2.18) we obtain (2.14). The conclusion thus comes from Theorem 2.1. The proof is complete. □

For this case where $H := H(t-s) \in \Psi$, we have that $h_1(t-s) = h_2(t-s)$. The subclass of Ψ containing such $H(t-s)$ is denoted by Ψ_0. Applying Theorem 2.1 to Ψ_0, we easily obtain the following result.

Theorem 2.3. *If for any $T \geqslant t_0$, there exist $H \in \Psi_0$, $j_0 \in I_m$, and $a, c \in R$ such that $T \leqslant a < c$ and*

$$\int_a^c \{H(s-a)(q_{j_0}(s) + q_{j_0}(2c-s)) - \frac{1}{4}h^2(s-a)[p(s-\sigma_{j_0}) + p(2c-s-\sigma_{j_0})]\}\,\mathrm{d}s > 0, \tag{2.19}$$

then every solution $u(x,t)$ of the problem (1.1), (1.2) *is oscillatory in G.*

Proof. Let $b = 2c - a$, then $H(b-c) = H(c-a) = H(b-a)/2$, and for any $w \in L[a,b]$, we have $\int_c^b w(s)\,\mathrm{d}s = \int_a^c w(2c-s)\,\mathrm{d}s$. Hence

$$\int_c^b H(b-s)q_{j_0}(s) = \int_c^b H(s-a)q_{j_0}(2c-s)\,\mathrm{d}s,$$

$$\int_c^b p(s-\sigma_{j_0})h^2(b-s)\,\mathrm{d}s = \int_a^c p(2c-s-\sigma_{j_0})h^2(s-a)\,\mathrm{d}s.$$

Thus that (2.19) holds implies that (2.14) holds for $H \in \Psi_0$, and therefore every solution of the problem (1.1), (1.2) is oscillatory in G by Theorem 2.1. The proof is complete. □

Let $H(t,s) = (t-s)^\lambda$, $t \geqslant s \geqslant t_0$, where $\lambda > 1$ is a constant. We obtain the following corollary.

Corollary 2.1. *If for each $r \geqslant t_0$ and for some $\lambda > 1$, and $j_0 \in I_m$,*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_r^t [(s-r)^\lambda q_{j_0}(s) - \frac{\lambda^2}{4}(s-r)^{\lambda-2}p(s-\sigma_{j_0})]\,\mathrm{d}s > 0, \tag{2.20}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_r^t [(t-s)^\lambda q_{j_0}(s) - \frac{\lambda^2}{4}(t-s)^{\lambda-2}p(s-\sigma_{j_0})]\,\mathrm{d}s > 0, \tag{2.21}$$

then every solution $u(x,t)$ of the problem (1.1), (1.2) *is oscillatory in G.*

Let $R(t) = \int_{t_0}^t \mathrm{d}s/p(s-\sigma_{j_0})$, $H(t,s) = [R(t) - R(s)]^\lambda$, $t \geqslant t_0$, where $\lambda > 1$ is a constant. Using Theorem 2.2, we easily obtain the following theorem.

Theorem 2.4. *If for each $r \geqslant t_0$ and for some $\lambda > 1$, $j_0 \in I_m$,*

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_r^t (R(s) - R(t))^\lambda q_{j_0}(s)\,\mathrm{d}s > \frac{\lambda^2}{4(\lambda-1)}, \tag{2.22}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_r^t (R(t)-R(s))^{\lambda} q_{j_0}(s)\,\mathrm{d}s > \frac{\lambda^2}{4(\lambda-1)}, \tag{2.23}$$

then every solution $u(x,t)$ *of* (1.1), (1.2) *is oscillatory in* G.

Proof. Since $H(t,s)=(R(t)-R(s))^{\lambda}$, then

$$h_1(t,s)=\lambda(R(t)-R(s))^{(\lambda-2)/2}\frac{1}{p(t-\sigma_{j_0})},$$

$$h_2(t,s)=\lambda(R(t)-R(s))^{(\lambda-2)/2}\frac{1}{p(s-\sigma_{j_0})}.$$

Nothing that

$$\int_r^t p(s-\sigma_{j_0})h_1^2(s,r)\,\mathrm{d}s = \int_r^t \lambda^2(R(s)-R(r))^{\lambda-2}\frac{\mathrm{d}s}{p(s-\sigma_{j_0})} = \frac{\lambda^2}{\lambda-1}(R(t)-R(r))^{\lambda-1},$$

and

$$\int_r^t p(s-\sigma_{j_0})h_2^2(s,r)\,\mathrm{d}s = \int_r^t \lambda^2(R(t)-R(r))^{\lambda-2}\frac{\mathrm{d}s}{p(s-\sigma_{j_0})} = \frac{\lambda^2}{\lambda-1}(R(t)-R(r))^{\lambda-1},$$

$$\lim_{t\to\infty} \frac{1}{4R^{\lambda-1}(t)} \int_r^t p(s-\sigma_{j_0})h_1^2(s,r)\,\mathrm{d}s = \frac{\lambda^2}{4(\lambda-1)}, \tag{2.24}$$

$$\lim_{t\to\infty} \frac{1}{4R^{\lambda-1}(t)} \int_r^t p(s-\sigma_{j_0})h_2^2(t,s)\,\mathrm{d}s = \frac{\lambda^2}{4(\lambda-1)}. \tag{2.25}$$

From (2.22) and (2.24), we have

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_r^t \{(R(s)-R(r))^{\lambda} q_{j_0}(s) - \frac{1}{4}p(s-\sigma_{j_0})h_1^2(s,r)\}\,\mathrm{d}s$$
$$= \limsup_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_r^t (R(s)-R(r))^{\lambda} q_{j_0}(s)\,\mathrm{d}s - \frac{\lambda^2}{4(\lambda-1)} > 0,$$

i.e., (2.15) holds. Similarly, (2.23) implies that (2.16) holds. By Theorem 2.2 , every solution $u(x,t)$ of the problem (1.1), (1.2) is oscillatory in G. □

The following Theorems are immediate results from Lemmas 2.5 and 2.6.

Theorem 2.5. *Assume that for any* $T \geqslant t_0$, *there exist* $H \in \Psi$, $F \in C^1([t_0,\infty); R)$, $j_0 \in I_m$, *and* $a, b, c \in R$ *such that* $T \leqslant a < c < b$,

$$\frac{1}{H(c,a)} \int_a^c \{H(s,a)\chi(s) - \frac{1}{4}p(s-\sigma_{j_0})F(s)h_1^2(s,a)\}\,\mathrm{d}s$$
$$+ \frac{1}{H(b,c)} \int_c^b \{H(b,s)\chi(s) - \frac{1}{4}p(s-\sigma_{j_0})h_2^2(b,s)\}\,\mathrm{d}s > 0. \tag{2.26}$$

Then every solution $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

Theorem 2.6. *If for some $H \in \Psi$, $F \in C^1([t_0, \infty); R)$, $j_0 \in I_m$, and for each $r \geqslant t_0$,*

$$\limsup_{t\to\infty} \int_r^t \{H(s,r)\chi(s) - \frac{1}{4}p(s-\sigma_{j_0})F(s)h_1^2(s,r)\}\,\mathrm{d}s > 0,$$

and

$$\limsup_{t\to\infty} \int_r^t \{H(t,s)\chi(s) - \frac{1}{4}p(s-\sigma_{j_0})F(s)h_2^2(t,s)\}\,\mathrm{d}s > 0,$$

then every solution $u(x,t)$ of the problem (1.1), (1.2) *is oscillatory in G.*

Theorem 2.7. *If for each $T \geqslant t_0$, there exist $H \in \Psi_0$, $F \in C^1([t_0, \infty); R)$, $j_0 \in I_m$, and $a, c \in R$ such that $T \leqslant a < c$ and*

$$\int_a^c \{H(s-a)[\chi(s) + \chi(2c-s)] - \frac{1}{4}h^2(s-a)[p(s-\sigma_{j_0})F(s) + p(2c-s-\sigma_{j_0})F(2c-s)]\}\,\mathrm{d}s > 0.$$

Then every solution $u(x,t)$ of the problem (1.1), (1.2) *is oscillatory in G.*

Corollary 2.2. *If for each $r \geqslant t_0$ and for some $\lambda > 1$, $j_0 \in I_m$,*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_r^t \{(s-r)^\lambda \chi(s)\,\mathrm{d}s - \frac{\lambda^2}{4}(s-r)^{\lambda-2}p(s-\sigma_{j_0})F(s)\}\,\mathrm{d}s > 0$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_r^t (t-s)^\lambda \chi(s)\,\mathrm{d}s - \frac{\lambda^2}{4}(t-s)^{\lambda-2}p(s-\sigma_{j_0})F(s)\}\,\mathrm{d}s > 0,$$

then every solution $u(x,t)$ of (1.1), (1.2) *is oscillatory in G.*

Remark 2.1. Our results above are sharper than the previous results.

Example 2.1. Consider the partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \left(\frac{1}{t+\pi} + \frac{1}{(t+\pi)^2}\right)\Delta u(x,t) + \frac{1}{t+\pi}\Delta u(x, t - \frac{3}{2}\pi) - \frac{1}{(t+1)^2}u(x,t) - \frac{\alpha}{t^2}u(x, t-3\pi), \quad (x,t) \in (0,\pi) \times [1,\infty), \tag{2.27}$$

with the boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, \quad t \geqslant 1. \tag{2.28}$$

Here $p(t) = 1$, $N = 1$, $s = 1$, $m = 1$, $\rho_1(t) = \frac{3}{2}\pi$, $\sigma_1 = 3\pi$, $\alpha > 0$ is a constant,

$$a(t) = \frac{1}{t+\pi} + \frac{1}{(t+\pi)^2}, \quad a_1(t) = \frac{1}{t+\pi}, \quad q(t) = \frac{1}{(t+1)^2}, \quad q_1(t) = \frac{\alpha}{t^2}.$$

Let $R(t) = \int_1^t 1\,\mathrm{d}s = t - 1$, for $\lambda > 1$, $r \geqslant 1$,

$$\lim_{t\to\infty} \frac{1}{R^{\lambda-1}(t)} \int_r^t (R(s) - R(r))^\lambda q_1(s)\,\mathrm{d}s = \lim_{t\to\infty} \frac{\alpha(t-r)^\lambda}{(\lambda-1)(t-1)^\lambda} = \frac{\alpha}{\lambda-1}. \tag{2.29}$$

Next, we will prove that

$$\int_r^t (R(t) - R(s))^\lambda q_1(s)\,\mathrm{d}s \geqslant \int_r^t (R(s) - R(r))^\lambda q_1(s)\,\mathrm{d}s. \tag{2.30}$$

Setting

$$G(t)=\int_r^t\{(R(t)-R(s))^{\lambda}-(R(s)-R(r))^{\lambda}\}q_1(s)\,\mathrm{d}s.$$

Then $G(r)=0$, and for $t\geqslant r$, we have

$$\begin{aligned}G'(t)&=\int_r^t\lambda(R(t)-R(s))^{\lambda-1}\frac{\alpha}{s^2}\,\mathrm{d}s-(R(t)-R(r))^{\lambda}\frac{\alpha}{t^2}\\&\geqslant\frac{\alpha}{t^2}\int_r^t\lambda(R(t)-R(s))^{\lambda-1}\,\mathrm{d}s-\frac{\alpha(R(t)-R(r))^{\lambda}}{t^2}\\&=\frac{\alpha}{t^2}(R(t)-R(r))^{\lambda}-\frac{\alpha(R(t)-R(r))^{\lambda}}{t^2}=0.\end{aligned}$$

Hence $G(t)\geqslant G(r)=0$, for $t\geqslant r$, i.e., (2.30) holds. By (2.29) and (2.30) for any $\alpha>\frac{1}{4}$, there exists $\lambda_0>1$ such that $\alpha/(\lambda_0-1)>\lambda_0^2/(4(\lambda_0-1))$. This means that (2.22) and (2.23) holds for the some $\lambda_0>1$. Applying Theorem 2.4, we have that every solution $u(x,t)$ of (2.27), (2.28) is oscillatory in $(0,\pi)\times[1,\infty)$ for $\alpha>\frac{1}{4}$. Because of the property $\int_1^{\infty}q_1(t)\,\mathrm{d}t=\int_1^{\infty}(\alpha/t^2)\,\mathrm{d}t<\infty$, the results of [3–5,9,10] can not be applied.

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